24 The Critical Function

1. Explain what is neutral geometry. What is deifference between Pasch geometry, protractor geometry and neutral geometry?

2. Let ℓ and ℓ' be two lines in neutral geometry. Show that, if ℓ and ℓ' possess a common perpendicular then $\ell || \ell'$.

3. Let ℓ be a line in a neutral geometry and let $P \notin \ell$. Let D be the foot of the perpendicular from P to ℓ. (i) If C is on the same side of line ℓ as P such that $m(\measuredangle DPC) = 90$ show that then $\overrightarrow{PC} \cap \ell = \emptyset$. (ii) If $m(\measuredangle DPC) > 90$ show that then $\overrightarrow{PC} \cap \ell = \emptyset$.

<u>**Definition**</u> **(least upper bound).** If $\mathcal{B}$ is a set of real numbers, then $r \in \mathcal{B}$ is a least upper bound of $\mathcal{B}$ (written $r = \text{lub}\mathcal{B}$) if (i) $b \le r$ for all $b \in \mathcal{B}$; and (ii) if $s < r$ then there is an element $b_s \in \mathcal{B}$ with $s < b_s$.

4. Find the least upper bound for each of the sets: (i) $\mathcal{B}_1 = \{-\frac{1}{n} | n \in \mathcal{N}\}$; (ii) $\mathcal{B}_2 = \{\sin(x) | x \in \mathbb{R}\}$; (iii) $\mathcal{B}_3 = \{x \in \mathbb{R} | x^3 < 2\}$.

<u>**Definition**</u> **(critical number $r(P,\ell)$).** Let ℓ be a line in a neutral geometry and let $P \notin \ell$. If D is the foot of the perpendicular from P to ℓ let $K(P,\ell) = \{r \in \mathbb{R} |$ there is a ray $\overrightarrow{PC}$ with $\overrightarrow{PC} \cap \ell \neq \emptyset$ and $r = m(\measuredangle DPC)\}$. The critical number for P and I is $r(P,\ell) = \text{lub}K(P,\ell)$.

5. Let ℓ be a line in a neutral geometry and let $P \notin \ell$. Let D be the foot of the perpendicular from P to ℓ. (i) If $m(\measuredangle DPC) = r(P,\ell)$ show that then $\overleftrightarrow{PC} || \ell$. (ii) If $m(\measuredangle DPC) > r(P,\ell)$ show that then $\overrightarrow{PC} \cap \ell = \emptyset$.

6. Let $P(a,b) \in \mathbb{H}$ with $a > 0$. If $\ell = {}_0L$, find $r(P,\ell)$.

<u>**Definition**</u> **(critical function $\Pi(t)$).** The critical function of a neutral geometry is the function $\Pi : \{t | t \ge 0\} \longrightarrow \{r | 0 \le r \le 90\}$ given by $\Pi(t) = r(P,\ell)$ where ℓ is any line and P is any point whose distance from ℓ is t.

7. Prove that in the Euclidean Plane $r(P,\ell) = 90$ for every line ℓ and every point $P \notin \ell$. Hence $\Pi(t) = 90$ for all t.

<u>**Definition**</u> **(HPP).** A neutral geometry satisfies the Hyperbolic Parallel Property (HPP) if for each line ℓ and each point $P \notin \ell$ there is more than one line through P parallel to ℓ.

<u>**Definition**</u> **(Euclidean geometry, hyperbolic geometry).** A Euclidean geometry is a neutral geometry that satisfies EPP. A hyperbolic geometry is a neutral geometry that satisfies HPP.

8. Prove that $(\mathbb{R}^2, \mathcal{L}_E, d_E, m_E)$ is a Euclidean geometry.

9. Prove that $(\mathbb{H}, \mathcal{L}_H, d_H, m_H)$ is a hyperbolic geometry.

IMPORTANT RESULTS (The Critical Function)

(24.1) Let ℓ be a line in a neutral geometry and let $P \notin \ell$. Let D be the foot of the perpendicular from P to ℓ. Then $\overrightarrow{PC} \cap \ell = \emptyset$, whenever $m(\measuredangle DPC) \ge 90$.

(24.2) In a neutral geometry and let $P \notin \ell$ and let D be the foot of the perpendicular from P to ℓ. If $m(\measuredangle DPC) \ge r(P,\ell)$ then $\overrightarrow{PC} \cap \ell = \emptyset$. If $m(\measuredangle DPC) < r(P,\ell)$ then $\overrightarrow{PC} \cap \ell \neq \emptyset$.

(24.3) Let ℓ be a line in a neutral geometry and P be a point not on ℓ. Then there is more than one line through P parallel to ℓ if and only if $r(P,\ell) < 90$.

(24.4) Let P and P' be points in a neutral geometry and let ℓ and ℓ' be lines with $P \notin \ell$ and $P' \notin \ell'$. If $d(P,\ell) = d(P',\ell')$ then $r(P,\ell) = r(P',\ell')$.

(24.5) In a neutral geometry, the critical function is nonincreasing, i.e., if $t' > t$ then $\Pi(t') \le \Pi(t)$.

(24.6) In a neutral geometry, if $\Pi(a) < 90$ then $\Pi(a/2) < 90$.

(24.7) In a neutral geometry, if $\Pi(a) < 90$ for some real number a, then $\Pi(a) < 90$ for all $t > 0$.

(24.8) **(All or None Theorem.)** In a neutral geometry, if there is one line ℓ' and one point $P' \notin \ell'$ such that there is a unique line through P' parallel to ℓ', then EPP holds.